OPTIMAL CONTROL

An Introduction to the Theory and Its Applications

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Table 5-1 Summary of Problems and of

	Problem				
	System	Cost	Time	Target set	Hamiltonian
1				$\mathbf{x}_1$ fixed end point	
2		$L = L(\mathbf{x}, \mathbf{u}),\ K = 0$	t_1 free	S_1 k-fold in R_n; $g_i(\mathbf{x}) = 0$ $i = 1, 2, \ldots, n-k$	
3	$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$	$L = L(\mathbf{x}, \mathbf{u}),\ K = K(\mathbf{x})$		R_n free end point	$H = H(\mathbf{x}, \mathbf{p}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \langle \mathbf{p}, \mathbf{f}(\mathbf{x},\mathbf{u})\rangle$
4		$L = L(\mathbf{x}, \mathbf{u}),\ K = 0$	t_1 fixed	$\mathbf{x}_1$ fixed end point	
5		$L = L(\mathbf{x}, \mathbf{u}),\ K = K(\mathbf{x})$		S_1 k-fold in R_n; $g_i(\mathbf{x}) = 0$ $i = 1, 2, \ldots, n-k$	
6				$\mathbf{x}_1$ fixed end point	
7		$L = L(\mathbf{x}, \mathbf{u}, t),\ K = 0$		$\mathbf{g}(t)$ moving point	
8			t_1 free	S $(k+1)$-fold in $R_n \times (T_1, T_2)$; $g_i(\mathbf{x}, t) = 0$ $i = 1, 2, \ldots, n-k$	
9	$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$	$L = L(\mathbf{x}, \mathbf{u}, t),\ K = K(\mathbf{x}, t)$		R_n free end point	$H = H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = L(\mathbf{x}, \mathbf{u}, t) + \langle \mathbf{p}, \mathbf{f}(\mathbf{x}, \mathbf{u}, t)\rangle$
10				$\mathbf{x}_1$ fixed end point	
11		$L = L(\mathbf{x}, \mathbf{u}, t),\ K = 0$	t_1 fixed	S_1 k-fold in R_n; $g_i(\mathbf{x}) = 0$ $i = 1, 2, \ldots, n-k$	
12		$L = L(\mathbf{x}, \mathbf{u}, t)$, $K = K(\mathbf{x})$		R_n free end point	
	A	B	C	D	E

Theorem 5-6P The Minimum Principle for Special Problem 2 *Let* $\mathbf{u}^*(t)$ *be an admissible control which transfers* $(\mathbf{x}_0, t_0)$ *to* $S = S_1 \times (T_1, T_2)$. *Let* $\mathbf{x}^*(t)$ *be the trajectory [of Eq. (5-388)] corresponding to* $\mathbf{u}^*(t)$, *originating at* $(\mathbf{x}_0, t_0)$, *and meeting* S *at* t_1. *In order that* $\mathbf{u}^*(t)$ *be optimal (for special problem 2), it is necessary that there exist a nonnegative constant* p_0^* *(that is,* $p_0^* \geq 0$*) and a function* $\mathbf{p}^*(t)$ *such that:*

a. $\mathbf{p}^*(t)$ *and* $\mathbf{x}^*(t)$ *are a solution of the canonical system*

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \mathbf{p}} [\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t), p_0^*] \tag{5-494}$$

$$\dot{\mathbf{p}}^*(t) = -\frac{\partial H}{\partial \mathbf{x}} [\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t), p_0^*] \tag{5-495}$$

satisfying the boundary conditions

$$\mathbf{x}^*(t_0) = \mathbf{x}_0 \qquad \mathbf{x}^*(t_1) \in S_1 \tag{5-496}$$

the Corresponding Necessary Conditions

Necessary conditions (1)	(2)	(3)	(4)	Theorem
$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}}\Big\vert_*$ $\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}}\Big\vert_*$	$H^* = H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*) = \min_{\mathbf{u}\in\Omega} H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u})$ or $H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*) \le H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u})$ for all $\mathbf{u}$ in Ω	$H^*[t] = H^*[t_1] = 0$	No condition on $\mathbf{p}^*(t_1)$	5-5
			$\mathbf{p}^*(t_1)$ normal to S_1 at $\mathbf{x}^*(t_1)$ $\mathbf{p}^*(t_1) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)}$	5-6
			$\mathbf{p}^*(t_1) = \frac{\partial K}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)}$	5-6 5-11
		$H^*[t] = H^*[t_1] = \text{const}$	No condition on $\mathbf{p}^*(t_1)$	5-10
			$\mathbf{p}^*(t_1) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)} + \frac{\partial K}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)}$	5-10 5-11
$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}}\Big\vert_*$ $\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}}\Big\vert_*$	$H^* = H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*, t) = \min_{\mathbf{u}\in\Omega} H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}, t)$ or $H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*, t) \le H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}, t)$ for all $\mathbf{u}$ in Ω	$H^*[t] = -\int_t^{t_1} \frac{\partial H}{\partial t}\Big\vert_* d\tau$ $H^*[t_1] = 0$	No condition on $\mathbf{p}^*(t_1)$	5-7
		$H^*[t] = \left\langle \mathbf{p}^*(t_1), \frac{d\mathbf{g}}{dt}\Big\vert_{*t_1} \right\rangle - \int_t^{t_1} \frac{\partial H}{\partial t}\Big\vert_* d\tau$ $H^*[t_1] = \left\langle \mathbf{p}^*(t_1), \frac{d\mathbf{g}}{dt}\Big\vert_{*t_1} \right\rangle$		5-8
		$H^*[t] = H^*[t_1] - \int_t^{t_1} \left[\frac{\partial H}{\partial t}\Big\vert_* + \frac{\partial^2 K}{\partial t^2}\Big\vert_* \right] d\tau$ $H^*[t_1] = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial t}\Big\vert_{*t_1} - \frac{\partial K}{\partial t}\Big\vert_{*t_1}$	$\mathbf{p}^*(t_1) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial \mathbf{x}}\Big\vert_{*t_1} + \frac{\partial K}{\partial \mathbf{x}}\Big\vert_{*t_1}$	5-11 5-9
		$H^*[t] = H^*[t_1] - \int_t^{t_1} \left[\frac{\partial H}{\partial t}\Big\vert_* + \frac{\partial^2 K}{\partial t^2}\Big\vert_* \right] d\tau$ $H^*[t_1] = -\frac{\partial K}{\partial t}\Big\vert_{*t_1}$	$\mathbf{p}^*(t_1) = \frac{\partial K}{\partial \mathbf{x}}\Big\vert_{*t_1}$	5-11 5-9
		$H^*[t] = H^*[t_1] - \int_t^{t_1} \frac{\partial H}{\partial t}\Big\vert_* d\tau$	No condition on $\mathbf{p}^*(t_1)$	5-10
			$\mathbf{p}^*(t_1)$ normal to S_1 at $\mathbf{x}^*(t_1)$ $\mathbf{p}^*(t_1) = \sum_{i=1}^{n-k} \alpha_i \frac{\partial g_i}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)}$	5-10
			$\mathbf{p}^*(t_1) = \frac{\partial K}{\partial \mathbf{x}}\Big\vert_{\mathbf{x}^*(t_1)}$	5-10 5-11
F	G	H	I	J

where the Hamiltonian function $H(\mathbf{x}, \mathbf{p}, \mathbf{u}, p_0)$ is given by

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, p_0) = p_0 L(\mathbf{x}, \mathbf{u}) + \langle \mathbf{p}, \mathbf{f}(\mathbf{x}, \mathbf{u}) \rangle \tag{5-497}$$

b. *The function $H[\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}, p_0^*]$ has an absolute minimum as a function of $\mathbf{u}$ over Ω at $\mathbf{u} = \mathbf{u}^*(t)$ for t in $[t_0, t_1]$; that is,*

$$\min_{\mathbf{u}\in\Omega} H[\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}, p_0^*] = H[\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t), p_0^*] \tag{5-498}$$

c. *The function $H[\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t), p_0^*]$ is zero for t in $[t_0, t_1]$; that is,*

$$H[\mathbf{x}^*(t), \mathbf{p}^*(t), \mathbf{u}^*(t), p_0^*] = 0 \qquad t \in [t_0, t_1] \tag{5-499}$$

d. *If S_1 is a smooth k-fold in R_n, then the vector $\mathbf{p}^*(t_1)$ is transversal (normal) to S_1 at $\mathbf{x}^*(t_1)$ (see Sec. 3-13); if $S_1 = R_n$, then $\mathbf{p}^*(t_1)$ is the zero vector; that is, $\mathbf{p}^*(t_1) = \mathbf{0}$.*